YOUNG MATHEMATICIANS AT WORK

CONSTRUCTING FRACTIONS, DECIMALS, AND PERCENTS

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“MATHEMATICS” OR “MATHEMATIZING”?

The United States suffers from “innumeracy” in its general population, “math avoidance” among high school students, and 50 percent failure among college calculus students. Causes include starvation budgets in the schools, mental attrition by television, parents [and teachers] who don’t like math. There’s another, unrecognized cause of failure: misconception of the nature of mathematics. . . . It’s the questions that drive mathematics. Solving problems and making up new ones is the essence of mathematical life. If mathematics is conceived apart from mathematical life, of course it seems—dead.

—Reuben Hersh

The mathematician’s best work is art, a high perfect art, as daring as the most secret dreams of imagination, clear and limpid. Mathematical genius and artistic genius touch one another.

—Gösta Mittag-Leffler

It is a truism that the purpose of teaching is to help students learn. Yet in the past teaching and learning were most often seen as two separate, even polar, processes. Teaching was what teachers did. They were supposed to know their subject matter and be able to explain it well. Students were supposed to do the learning. They were expected to work hard, practice, and listen to understand. If they didn’t learn, it was their fault; they had a learning disability, they needed remediation, they were preoccupied, or they were lazy. Even when we spoke of development, it was usually in connection with assessing learners to see whether they were developmentally ready for the teacher’s instruction.

Interestingly, in some languages, learning and teaching are the same word. In Dutch, for example, the distinction between learning and teaching is made only by the preposition. The verb is the same. Leren aan means teaching; leren van means learning. When learning and teaching are so closely related, they will be integrated in learning/teaching frameworks: teaching will be seen as closely related to learning, not only in language and thought but also in action. If learning doesn’t happen, there has been no teaching. The actions of learning and teaching are inseparable.
Of course, different teachers have different styles of helping children learn. But behind these styles are frameworks based on teachers’ beliefs about the learning/teaching process. These frameworks, in turn, affect how teachers interact with children, what questions they ask, what ideas they pursue, and even what activities they design or select. Teachers make many important decisions—some of them in a split second in the nitty-gritty of the classroom. In making these decisions, some teachers are led by the structure of mathematics or the textbook, others by the development of the children.

**LEARNING AND TEACHING IN THE CLASSROOM**

Join us in Carol Mosesson’s fourth/fifth-grade classroom, in New Rochelle, New York. She is telling her students about a dilemma that occurred in her class the previous year and how she wants to be sure it doesn’t happen again.*

“Last year,” Carol explains, “I took my students on field trips related to the projects we were working on. At one point, we went to several places in New York City to gather research. I got some parents to help me, and we scheduled four field trips in one day. Four students went to the Museum of Natural History, five went to the Museum of Modern Art, eight went with me to Ellis Island and the Statue of Liberty, and the five remaining students went to the Planetarium. The problem we ran into was that the school cafeteria staff had made seventeen submarine sandwiches for the kids for lunch. They gave three sandwiches to the four kids going to the Museum of Natural History. The five kids in the second group got four subs. The eight kids going to Ellis Island got seven subs, and that left three for the five kids going to the Planetarium.” As she talks, she draws a picture (see Figure 1.1) on chart paper of the context she is developing. “Now we didn’t eat together, obviously, because we were all in different parts of the city. The next day after talking about our trips, several of the kids complained that it hadn’t been fair, that some kids got more to eat. What do you think about this? Were they right? Because if they were, I would really like to work out a fair system—one where I would know how many subs to give each group when we go on field trips this year.”

Carol is introducing fair sharing—a rich, real context in which her students can generate and model for themselves mathematical ideas related to fractions. When children are given trivial word problems, they often just ask themselves what operation is called for; the context becomes irrelevant as they manipulate numbers, applying what they know. Truly problematic contexts engage children in a way that keeps them grounded. They attempt to model the situation mathematically, as a way to make sense of it. They notice patterns, raise conjectures, and then defend them to one another.

*Although this investigation was embellished and situated in the context of field trips by Carol Mosesson, the kernel of the activity—fair sharing of submarine sandwiches—comes from Mathematics in Context, Encyclopaedia Britannica Educational Corporation.
“Turn to the person you are sitting next to and talk for a few minutes about whether you think this situation was fair,” Carol continues.

“It’s fair,” several children comment. “It’s one less sandwich than kids each time. It’s three subs for four kids, so when there are five kids, they gave them four. For eight kids, they gave them seven.”

“Yeah, but it wasn’t fair for the Planetarium group. They had five kids and they only got three subs.” Jackie is adamant as she comes to their defense. “The Museum of Modern Art group had five kids, too, and they got four sandwiches!”

“But you could just cut them in different pieces, like fourths or fifths,” John offers.

“But the pieces would be different sizes. It’s not fair.”

“What do you think about her argument, John? Did everyone get the same amount?” Carol inquires, interested in whether John thinks the pieces would be equivalent. He shakes his head, acknowledging that the pieces would not be the same size. Another student raises his hand. “Michael?”

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FIGURE 1.1 Subs for the Field Trip
“But the others are all the same.” As Michael advances this common misconception, several of his classmates nod in agreement. “You keep adding one child and one sub. Three for four is the same as four for five.”

“If the Planetarium group had another sub it would all be the same,” Aysha says, verbalizing what most are thinking.

Carol has succeeded in making their initial ideas visible. Now she wants to create disequilibrium. “Well, let’s investigate this some more. I suggest we work with our math workshop partners and check out two things.” She writes the questions down on a large piece of chart paper as she talks. “How much did each child in each group get, assuming the subs were all shared equally in each group? And which group got the most?”

Several students continue to voice their agreement with Aysha and Michael. “I think they all got the same except for the Planetarium group.”

“Well, let’s check it out and see. Let’s investigate a bit, prove your ideas, and we’ll have a math congress after we’ve worked on it some. Use any materials you want, or make drawings to help you prove your thinking. We’ll share our thinking when everyone feels he or she has had enough time to work on this.”

The children set to work investigating the problem. Some take out Unifix cubes; most draw the subs and show how they would cut them. Carol moves from group to group, ensuring that everyone is busy and clear about the problem. She sits down with Jackie and Ernie for a moment.

“There were three subs for four kids,” Jackie explains, “so we cut two subs in half, then the last in fourths. So everybody in that group got one half plus one fourth.”

Ernie, her partner, explains their work for the Ellis Island group. “And we’re doing the same thing here. See—we’re giving each person a half sub first. That is four subs. One sub we’re cutting into eight pieces, and the other two into fourths.”

“So how much does each person in the group get?” Carol inquires.

“One half plus one eighth plus one fourth.”

“And you’re going to continue like this with the Planetarium and the Museum of Modern Art groups?” Jackie and Ernie nod yes. Carol notes their strategy: they are making unit fractions—fractions with numerators of one.

Nicole and Michelle, whom she visits next, have worked on the three-sub-four-kids situation very similarly. But for the four-subs-five-kids situation they have changed their strategy. Michelle explains, “We divided these subs up into fifths, because there were five kids.”

Carol sees that each of the four subs is drawn and cut up into fifths. “So how much of a sub did each child get?” she asks.

“One of these, one of these, one of these, and one of these.” Nicole points to a fifth from each sub. “That’s four fifths of a sub for each kid because it is four times one fifth.”

“Oh, that’s interesting, isn’t it? Four subs for five kids ended up being four fifths of a sub.” Carol attempts to point up the relationship, but Nicole and Michelle seem uninterested, and they go back to using unit fractions for
the seven-sub-eights-kids problem. Carol debates whether or not to suggest that they try the same strategy they used for the four-sub-fives-kids situation but decides against it. They are very involved in the context and they (as well as Jackie and Ernie) are cutting up the subs in ways that make sense *within the context*. They can imagine subs cut up into halves, fourths, or even fifths. These are reasonable sizes. Eighths are not, unless they become necessary at the end to share a last sub fairly. The realistic nature of the context enables children to realize what they are doing, to check whether it makes sense.

Working within a context also develops children’s ability to make mathematical meaning of their own many lived worlds. There is much to be investigated yet. Asking children to adopt multiplication and division shortcuts too soon may actually impede genuine learning. As the well-known mathematician George Pólya (1887–1985) once pointed out, “When introduced at the wrong time or place, good logic may be the worst enemy of good teaching.” Historically (see the in-depth discussion in Chapter 3), unit fractions were used for a very long time before common fractions were accepted. As the children explore fair-sharing situations with unit fractions, they will have experiences comparing and making equivalent fractions. These experiences will bring up some big ideas and support the development of some powerful mathematical strategies. Let’s return to Jackie and Ernie at work to witness this process.

Jackie has drawn four subs, and she and Ernie are deciding how to share this amount fairly among five kids—the ones who went to the Museum of Modern Art. “Let’s give everyone a half again first. That leaves one and a half subs left.”

“So let’s cut the whole one up into fifths.” Ernie draws four lines, making five equal parts, as he talks.

“Okay, so now every kid has one half plus one fifth,” Jackie continues, “and now we have to cut up the last half into fifths. So it’s one half plus one fifth plus—one fifth? That can’t be—these are just slivers.” She points to the small pieces resulting from cutting up the half sub. “These slivers are only about half the size of those fifths. So what do we call these?”

Jackie and Ernie are struggling with a big idea. To resolve their puzzlement they must grapple with the heart of fractional relationships. One fifth of a half sub is smaller than one fifth of a whole sub. What is the whole? And what is the equivalence of this piece to the whole?

“You’re right—two of these fifths [the slivers] are about the same as one of those fifths,” Ernie ponders. “So maybe it’s a half of a fifth. But what is that?”

“We know it is a fifth of a half of a sub—”

“Yeah, but it is also one half of a fifth—see.” Ernie points out the relationship of the sliver to the fifths that resulted when one sub was cut into five equal pieces. He is noticing that \( \frac{1}{2} \times \frac{1}{5} = \frac{1}{5} \times \frac{1}{2} \). This relationship is a specific example of the commutative property of multiplication, although for him at this point it is probably not generalizable.

“Well, if it’s a half of a fifth, then it must be a tenth,” Jackie offers.